

A PROOF OF SOME CONJECTURES OF MAO ON PARTITION RANK INEQUALITIES

ETHAN ALWAISE, ELENA IANNUZZI, AND HOLLY SWISHER

ABSTRACT. Based on work of Atkin and Swinnerton-Dyer on partition rank difference functions, and more recent work of Lovejoy and Osburn, Mao has proved several inequalities between partition ranks modulo 10, and additional results modulo 6 and 10 for the M_2 rank of partitions without repeated odd parts. Mao conjectured some additional inequalities. We prove some of Mao's rank inequality conjectures for both the rank and the M_2 rank modulo 10 using elementary methods.

1. INTRODUCTION AND STATEMENT OF RESULTS

For a positive integer n , a *partition* of n is a non-increasing sequence of positive integers that sum to n , where each summand is called a *part*. The partition function $p(n)$ counts the number of partitions of n , and we define $p(0) = 1$.

The celebrated Ramanujan congruences demonstrate compelling divisibility properties for $p(n)$,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Dyson [3] defined the *rank* of a partition λ to be $l(\lambda) - n(\lambda)$, where $l(\lambda)$ and $n(\lambda)$ denote the largest part and number of parts of λ , respectively. Dyson conjectured that this gave a combinatorial explanation for the Ramanujan congruences modulo 5 and 7. In particular, if $N(s, m, n)$ is defined to be the number of partitions of n that have rank congruent to s modulo m , then Dyson conjectured that for each residue class s ,

$$\begin{aligned} (1) \quad N(s, 5, 5n+4) &= \frac{p(5n+4)}{5} \\ (2) \quad N(s, 7, 7n+5) &= \frac{p(7n+5)}{7}. \end{aligned}$$

Atkin and Swinnerton-Dyer [2] proved (1), (2) by obtaining generating functions for rank differences of the form $N(s, \ell, \ell n + b) - N(t, \ell, \ell n + b)$ for $\ell = 5, 7$, and showing that the relevant differences were always 0 in the setting $(\ell, b) \in \{(5, 4), (7, 5)\}$. They determined all of the generating functions for $N(s, \ell, \ell n + b) - N(t, \ell, \ell n + b)$ where $\ell = 5, 7$, and obtained several interesting identities for the non-Ramanujan cases.

Lovejoy and Osburn [4, 5, 6] used similar techniques to obtain interesting generating function representations for rank differences of overpartitions, as well as partitions without repeated odd parts. For example, let λ be a partition without repeated odd parts. The M_2 rank of λ is defined to be

$$\left\lceil \frac{l(\lambda)}{2} \right\rceil - n(\lambda).$$

Date: September 22, 2015.

2010 Mathematics Subject Classification. 11P83.

Key words and phrases. partitions, ranks, rank differences.

This work was supported by the National Science Foundation REU Grant DMS-1359173.

Let $N_2(s, m, n)$ count the number of partitions of n with distinct odd parts and M_2 rank congruent to s modulo m . Lovejoy and Osburn [5] obtained generating function identities for rank differences of the form $N_2(s, \ell, \ell n + b) - N_2(t, \ell, \ell n + b)$ for $\ell = 3$ and $\ell = 5$.

Most recently, Mao [7, 8] has derived generating function formulas for Dyson's rank on partitions modulo 10, and the M_2 rank on partitions without repeated odd parts modulo 6 and 10. In this work he proves a number of inequalities, including for example

$$\begin{aligned} N(0, 10, 5n + 1) &> N(4, 10, 5n + 1), \\ N_2(0, 6, 3n) + N_2(1, 6, 3n) &> N_2(2, 6, 3n) + N_2(3, 6, 3n). \end{aligned}$$

Mao gives the following conjectures based on computational evidence. The first, is for Dyson's rank on unrestricted partitions.

Conjecture 1.1. *Computational evidence suggests that*

- (3) $N(0, 10, 5n) + N(1, 10, 5n) > N(4, 10, 5n) + N(5, 10, 5n)$ for $n \geq 0$,
- (4) $N(1, 10, 5n) + N(2, 10, 5n) \geq N(3, 10, 5n) + N(4, 10, 5n)$ for $n \geq 1$.

The second, is for the M_2 rank on partitions without repeated odd parts.

Conjecture 1.2. *Computational evidence suggests that*

- (5) $N_2(0, 10, 5n) + N_2(1, 10, 5n) > N_2(4, 10, 5n) + N_2(5, 10, 5n)$ for $n \geq 0$,
- (6) $N_2(0, 10, 5n + 4) + N_2(1, 10, 5n + 4) > N_2(4, 10, 5n + 4) + N_2(5, 10, 5n + 4)$ for $n \geq 0$,
- (7) $N_2(1, 10, 5n) + N_2(2, 10, 5n) > N_2(3, 10, 5n) + N_2(4, 10, 5n)$ for $n \geq 1$,
- (8) $N_2(1, 10, 5n + 2) + N_2(2, 10, 5n + 2) > N_2(3, 10, 5n + 2) + N_2(4, 10, 5n + 2)$ for $n \geq 1$,
- (9) $N_2(0, 6, 3n + 2) + N_2(1, 6, 3n + 2) > N_2(2, 6, 3n + 2) + N_2(3, 6, 3n + 2)$ for $n \geq 0$.

In this paper we prove the following theorem using elementary techniques.

Theorem 1.3. *Mao's conjectures (3), (4), (5), and (6) are true. In fact, in (4), the strict inequality holds.*

We note that our method did not suffice to prove the remaining three conjectures, which are still open.

The rest of the paper is organized as follows. In Section 2, we gather some definitions, notation, and lemmas that will be used later. In Section 3, we prove Theorem 1.3.

2. PRELIMINARIES

We use the following standard q -series notation. For $n \in \mathbb{N}$, $a \in \mathbb{C}$, define

$$\begin{aligned} (a; q)_n &:= \prod_{i=0}^{n-1} (1 - aq^i) \\ (a; q)_\infty &:= \prod_{i=0}^{\infty} (1 - aq^i), \end{aligned}$$

and also define $(a; q)_0 = 1$. As shorthand, write

$$\begin{aligned} (a_1, \dots, a_k; q)_n &:= (a_1; q)_n \cdots (a_k; q)_n \\ (a_1, \dots, a_k; q)_\infty &:= (a_1; q)_\infty \cdots (a_k; q)_\infty. \end{aligned}$$

Furthermore, we will make use of the following notation of Mao.¹ For positive integers $a < b$, define

$$\begin{aligned} J_b &:= (q^b; q^b)_\infty \\ J_{a,b} &:= (q^a, q^{b-a}, q^b; q^b)_\infty, \\ L_{a,b} &:= \frac{J_b^2}{J_{a,b}}. \end{aligned}$$

Lemma 2.1 (Mao [7]). *Given positive integers $a < b$, the q -series coefficients of $L_{a,b}$ are all nonnegative.*

Mao proved rank difference formulas that we will use in our proof of Theorem 1.3. First, for unrestricted partitions, Mao proved the following theorem.

Theorem 2.2 (Mao [7]). *We have that*

$$\begin{aligned} & \sum_{n=0}^{\infty} (N(0, 10, n) + N(1, 10, n) - N(4, 10, n) - N(5, 10, n)) q^n \\ &= \left(\frac{J_{25} J_{50}^5 J_{20,50}^2}{J_{10,50}^4 J_{15,50}^3} + \frac{1}{J_{25}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{75n(n+1)/2+5}}{1 + q^{25n+5}} \right) + q \left(\frac{J_{25} J_{50}^5}{J_{5,50} J_{10,50}^2 J_{15,50}^2} \right) \\ &+ q^2 \left(\frac{J_{25} J_{50}^5}{J_{5,50}^2 J_{15,50} J_{20,50}^2} \right) + q^3 \left(\frac{J_{25} J_{50}^5 J_{10,50}^2}{J_{5,50}^3 J_{20,50}^4} - \frac{1}{J_{25}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{75n(n+1)/2+5}}{1 + q^{25n+10}} \right) \\ &+ q^4 \left(\frac{2J_{50}^6}{J_{25} J_{5,50} J_{10,50} J_{15,50} J_{20,50}} \right), \end{aligned}$$

and

$$\begin{aligned} & \sum_{n=0}^{\infty} (N(1, 10, n) + N(2, 10, n) - N(3, 10, n) - N(4, 10, n)) q^n \\ &= \left(\frac{2q^5 J_{50}^6}{J_{25} J_{10,50}^2 J_{15,50}^2} - \frac{1}{J_{25}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{75n(n+1)/2+5}}{1 + q^{25n+5}} \right) \\ &+ q \left(\frac{2q^5 J_{50}^6}{J_{25} J_{5,50} J_{15,50} J_{20,50}^2} \right) + q^2 \left(\frac{J_{25} J_{50}^5 J_{20,50}}{J_{10,50}^3 J_{15,50}^3} \right) + q^3 \left(\frac{J_{25} J_{50}^5}{J_{5,50} J_{10,50} J_{15,50}^2 J_{20,50}} \right) \\ &+ q^4 \left(\frac{J_{25} J_{50}^5 J_{20,50}^2 J_{25,50}}{2q^5 J_{10,50}^4 J_{15,50}^4} - \frac{1}{q^5 J_{25}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(75n^2+25n)/2}}{1 + q^{25n}} \right). \end{aligned}$$

We will also make use of the following theorem for M_2 rank of partitions without repeated odd parts.

Theorem 2.3 (Mao [8]). *We have that*

$$\sum_{n=0}^{\infty} (N_2(0, 10, n) + N_2(1, 10, n) - N_2(4, 10, n) - N_2(5, 10, n)) q^n$$

¹We note that our definition of $L_{a,b}$ differs from Mao's in that the roles of a and b are reversed.

$$\begin{aligned}
&= \left(\frac{2q^5 J_{100}^{15} J_{10,100} J_{50,100}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^2 J_{25,100}^3 J_{30,100} J_{35,100}^2 J_{45,100}^3} + \frac{1}{J_{25,100}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}} \right) \\
&+ q \left(\frac{J_{100}^{15} J_{20,100} J_{30,100}^2 J_{50,100}}{J_{5,100}^2 J_{10,100}^2 J_{15,100}^2 J_{25,100} J_{35,100}^4 J_{40,100}^3 J_{45,100}^2} \right) \\
&+ q^2 \left(\frac{J_{100}^{15} J_{50,100}}{J_{5,100}^3 J_{15,100}^3 J_{20,100} J_{25,100} J_{35,100}^3 J_{40,100} J_{45,100}^3} \right) \\
&+ q^3 \left(\frac{J_{100}^{15} J_{10,100}^2 J_{40,100} J_{50,100}}{J_{5,100}^4 J_{15,100}^2 J_{20,100}^3 J_{25,100} J_{30,100}^2 J_{35,100}^2 J_{45,100}^4} \right) \\
&+ q^4 \left(\frac{2J_{100}^{15} J_{30,100} J_{50,100}}{J_{5,100}^2 J_{10,100} J_{15,100}^3 J_{25,100}^3 J_{35,100}^3 J_{40,100}^2 J_{45,100}^2} + \frac{1}{J_{25,100}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+75n+20}}{1+q^{50n+30}} \right).
\end{aligned}$$

In addition, we require the following two facts about q -series which follow directly from the definitions. For integers a, b, c we have

$$(10) \quad (q^a; q^b)_{\infty} (-q^a; q^b)_{\infty} = (q^{2a}; q^{2b})_{\infty},$$

$$(11) \quad (cq^a; q^{2b})_{\infty} (cq^{a+b}; q^{2b})_{\infty} = (cq^a; q^b)_{\infty}.$$

Finally, we recall the Jacobi Triple Product formula, which can be found in [1],

$$(12) \quad \sum_{n \in \mathbb{Z}} z^n q^{n^2} = (-zq, -q/z, q^2; q^2)_{\infty}.$$

3. PROOF OF THEOREM 1.3

3.1. Proof of (3). In order to prove (3), we need to show that the series

$$\sum_{n=0}^{\infty} (N(0, 10, 5n) + N(1, 10, 5n) - N(4, 10, 5n) - N(5, 10, 5n)) q^n$$

has strictly positive coefficients. Using the first part of Theorem 2.2, we see that

$$\sum_{n=0}^{\infty} (N(0, 10, n) + N(1, 10, n) - N(4, 10, n) - N(5, 10, n)) q^n = S_0 + qS_1 + q^2S_2 + q^3S_3 + q^4S_4,$$

where each S_i is a series in q^5 . Thus we can obtain our desired generating function by letting $q \mapsto q^{\frac{1}{5}}$ in S_0 . We obtain that

$$\begin{aligned}
(13) \quad &\sum_{n=0}^{\infty} (N(0, 10, 5n) + N(1, 10, 5n) - N(4, 10, 5n) - N(5, 10, 5n)) q^n \\
&= \frac{J_5 J_{10}^5 J_{4,10}^2}{J_{2,10}^4 J_{3,10}^3} + \frac{1}{J_5} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} = \frac{1}{J_5} \left(\frac{J_5^2 J_{10}^5 J_{4,10}^2}{J_{2,10}^4 J_{3,10}^3} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} \right).
\end{aligned}$$

We will now show that (13) has strictly positive q -series coefficients for $n \geq 0$. Since $\frac{1}{J_5}$ has all nonnegative coefficients and a constant term of 1, it suffices to show that

$$\frac{J_5^2 J_{10}^5 J_{4,10}^2}{J_{2,10}^4 J_{3,10}^3} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}}$$

has all positive coefficients. First, we split the sum into nonnegative and negative indices, and reindex to see that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{(15n^2-5n)/2}}{1+q^{5n-1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1} (1-q^{5n+1})}{1-q^{10n+2}} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{(15n^2-5n)/2} (1-q^{5n-1})}{1-q^{10n-2}}. \end{aligned}$$

Now, we split according to the summation index n modulo 2, to obtain

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} &= \sum_{n=0}^{\infty} \frac{q^{(15(2n)^2+15(2n))/2+1} (1-q^{5(2n)+1})}{1-q^{10(2n)+2}} - \sum_{n=0}^{\infty} \frac{q^{(15(2n+1)^2+15(2n+1))/2+1} (1-q^{5(2n+1)+1})}{1-q^{10(2n+1)+2}} \\ &\quad + \sum_{n=1}^{\infty} \frac{q^{(15(2n)^2-5(2n))/2} (1-q^{5(2n)-1})}{1-q^{10(2n)-2}} - \sum_{n=1}^{\infty} \frac{q^{(15(2n-1)^2-5(2n-1))/2} (1-q^{5(2n-1)-1})}{1-q^{10(2n-1)-2}}. \end{aligned}$$

Gathering the positive summands together, we see that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} = S - T_1 - T_2 - T_3 - T_4,$$

where

$$S := \sum_{n=0}^{\infty} \frac{q^{30n^2+15n+1}}{1-q^{20n+2}} + \sum_{n=0}^{\infty} \frac{q^{30n^2+55n+22}}{1-q^{20n+12}} + \sum_{n=1}^{\infty} \frac{q^{30n^2-5n}}{1-q^{20n-2}} + \sum_{n=1}^{\infty} \frac{q^{30n^2-25n+4}}{1-q^{20n-12}},$$

and

$$\begin{aligned} T_1 &= \sum_{n=0}^{\infty} a_1(n) q^n := \sum_{n=0}^{\infty} \frac{q^{30n^2+25n+2}}{1-q^{20n+2}}, \\ T_2 &= \sum_{n=0}^{\infty} a_2(n) q^n := \sum_{n=0}^{\infty} \frac{q^{30n^2+45n+16}}{1-q^{20n+12}}, \\ T_3 &= \sum_{n=0}^{\infty} a_3(n) q^n := \sum_{n=1}^{\infty} \frac{q^{30n^2+5n-1}}{1-q^{20n-2}}, \\ T_4 &= \sum_{n=0}^{\infty} a_4(n) q^n := \sum_{n=1}^{\infty} \frac{q^{30n^2-35n+10}}{1-q^{20n-12}}. \end{aligned}$$

We see that S, T_1, \dots, T_4 all have nonnegative coefficients. Thus to prove (3), it suffices to show that

$$\frac{J_5^2 J_{10}^5 J_{4,10}^2}{J_{3,10}^3 J_{2,10}^4} - T_1 - T_2 - T_3 - T_4$$

has positive coefficients. Let $T_1 + T_2 + T_3 + T_4 = \sum_{n=1}^{\infty} a(n) q^n$, and let

$$\frac{J_5^2 J_{10}^5 J_{4,10}^2}{J_{3,10}^3 J_{2,10}^4} = 1 + \sum_{n=1}^{\infty} b(n) q^n.$$

We will show that $b(n) > a(n)$ for all $n \geq 1$.

Expanding the denominator of T_1 as a geometric series, we see that

$$T_1 = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q^{30n^2 + (20k+25)n + (2k+2)}.$$

Thus for a given $N \geq 0$, we see that $a_1(N)$ counts the number of nonnegative integer pairs (n, k) such that

$$(14) \quad N = 30n^2 + (20k + 25)n + (2k + 2).$$

Clearly for each choice of $n \geq 0$ there is at most one $k \geq 0$ such that (n, k) is a solution to (14). Also, since $(20k + 25)n + (2k + 2)$ is positive for all $n, k \geq 0$, if $n \geq \sqrt{\frac{N}{30}}$, then no solutions exist.

Thus, we have that $a_1(N) \leq \lfloor \sqrt{\frac{N}{30}} \rfloor + 1$ for all $N \geq 0$.

Similarly,

$$T_2 = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q^{30n^2 + (20k+45)n + (12k+16)},$$

and so $a_2(N) \leq \lfloor \sqrt{\frac{N}{30}} \rfloor + 1$ for all $N \geq 0$ as well. For T_3 , we have

$$T_3 = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{30n^2 + (20k+5)n - (2k+1)}.$$

Since the sum starts at $n = 1$ we have one fewer term. Also, we see that $(20k + 5)n - (2k + 1)$ is positive for all $n \geq 1, k \geq 0$. Thus we get a bound of $a_3(N) \leq \lfloor \sqrt{\frac{N}{30}} \rfloor$ for all $N \geq 1$. The T_4 case is a little different. Here,

$$T_4 = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{30n^2 + (20k-35)n - (12k-10)}.$$

We observe that $30n^2 + (20k - 35)n - (12k - 10) \geq 5(3n - 2)(2n - 1) \geq 5(2n - 1)^2$ for all $n \geq 1$. Thus if $2n - 1 \geq \sqrt{\frac{N}{5}}$, then no solutions exist to $N = 30n^2 + (20k - 35)n - (12k - 10)$. We then get a bound of $a_4(N) \leq \lfloor \sqrt{\frac{N}{20}} \rfloor + 1$ for all $N \geq 1$.

Together, noting that none of the T_i have a constant term, we see that for any $n \geq 1$,

$$(15) \quad a(n) \leq 3 \left\lfloor \sqrt{\frac{n}{30}} \right\rfloor + \left\lfloor \sqrt{\frac{n}{20}} \right\rfloor + 3.$$

By (10) and (11), we see that

$$\begin{aligned} \frac{J_5^2 J_{10}^5 J_{4,10}^2}{J_{2,10}^4 J_{3,10}^3} &= \frac{(q^5; q^5)_{\infty}^2 (q^4, q^6; q^{10})_{\infty}^2}{(q^3, q^7; q^{10})_{\infty}^3 (q^2, q^8; q^{10})_{\infty}^4} \\ &= \frac{(q^5; q^5)_{\infty}^2 (-q^2, -q^3, -q^7, -q^8; q^{10})_{\infty}^2}{(q^3, q^7; q^{10})_{\infty} (q^2, q^8; q^{10})_{\infty}^2} \\ &= \frac{(q^5; q^5)_{\infty}^2 (-q^2, -q^3; q^5)_{\infty}^2}{(q^3, q^7; q^{10})_{\infty} (q^2, q^8; q^{10})_{\infty}^2}. \end{aligned}$$

Applying (12) with $z = q^{1/2}$ and $q = q^{5/2}$, we obtain

$$\frac{J_5^2 J_{4,10}^2 J_{10}^5}{J_{3,10}^3 J_{2,10}^4} = \frac{1}{(q^3, q^7; q^{10})_{\infty} (q^2, q^8; q^{10})_{\infty}^2} \left[\sum_{n=-\infty}^{\infty} q^{n(5n+1)/2} \right]^2$$

$$= \frac{1}{(1-q^2)(1-q^3)} \cdot \frac{1}{(q^2, q^7, q^{12}, q^{13}; q^{10})_{\infty} (q^8; q^{10})_{\infty}^2} \left[\sum_{n=-\infty}^{\infty} q^{n(5n+1)/2} \right]^2,$$

where we observe that all series involved in this product have nonnegative coefficients. Let

$$\frac{1}{(1-q^2)(1-q^3)} = \sum_{i,j \geq 0} q^{2i+3j} = \sum_{n=0}^{\infty} c(n)q^n.$$

We note that $c(0) = 1$ and for $n \geq 1$, $c(n)$ is equal to the number of nonnegative integer solutions (i, j) of the equation $2i + 3j = n$. For a fixed $n \geq 1$, and $j \geq 0$, we see that there is at most one $i \geq 0$ for which (i, j) is a solution, and such an i exists if and only if $0 \leq j \leq n/3$ and $j \equiv n \pmod{2}$. Considering each possible residue of n modulo 6, we see that in all cases, $c(n) \geq \lfloor \frac{n}{6} \rfloor$. Thus, we have that for all $n \geq 1$,

$$(16) \quad b(n) \geq \left\lfloor \frac{n}{6} \right\rfloor.$$

It suffices then to show that $\frac{n}{6} > 3\sqrt{\frac{n}{30}} + \sqrt{\frac{n}{20}} + 4$ for sufficiently large n , and to check that $b(n) > a(n)$ for all remaining cases. We have that $\frac{n}{6} > 3\sqrt{\frac{n}{30}} + \sqrt{\frac{n}{20}} + 4$ if and only if $\frac{1}{6}n - (\frac{\sqrt{30}+\sqrt{5}}{10})\sqrt{n} - 4 \geq 0$, which occurs for $n \geq 60$. Moreover, we also see that $b(n) > a(n)$ for $1 \leq n \leq 59$, by a quick Maple calculation, which completes the proof of (3).

For the remaining conjectures we use a similar technique, so give a somewhat abbreviated discussion of the proofs.

3.2. Proof of (4). In order to prove (4), we need to show that

$$\sum_{n=1}^{\infty} (N(1, 10, 5n) + N(2, 10, 5n) - N(3, 10, 5n) - N(4, 10, 5n))q^n$$

has nonnegative coefficients. Using the second part of Theorem 2.2, we obtain that

$$(17) \quad \sum_{n=1}^{\infty} (N(1, 10, 5n) + N(2, 10, 5n) - N(3, 10, 5n) - N(4, 10, 5n))q^n \\ = \frac{1}{J_5} \left(\frac{2qJ_{10}^6}{J_{2,10}^2 J_{3,10}^2} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}} \right).$$

Since $\frac{1}{J_5}$ has all nonnegative coefficients and a constant term of 1, it suffices to show that

$$\frac{2qJ_{10}^6}{J_{2,10}^2 J_{3,10}^2} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(15n^2+15n)/2+1}}{1+q^{5n+1}}$$

has all nonnegative coefficients. We observe the sum in this case is the same as the sum in the proof of (3). However in this setting we are subtracting, rather than adding the sum. Thus by our dissection in the last subsection, it suffices to prove that

$$\frac{2qJ_{10}^6}{J_{2,10}^2 J_{3,10}^2} - T'_1 - T'_2 - T'_3 - T'_4$$

has positive coefficients, where

$$T'_1 = \sum_{n=0}^{\infty} a'_1(n)q^n := \sum_{n=0}^{\infty} \frac{q^{30n^2+15n+1}}{1-q^{20n+2}} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q^{30n^2+15n+1+k(20n+2)},$$

$$\begin{aligned}
T'_2 &= \sum_{n=0}^{\infty} a'_2(n)q^n := \sum_{n=0}^{\infty} \frac{q^{30n^2+55n+22}}{1-q^{20n+12}} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q^{30n^2+55n+22+k(20n+12)}, \\
T'_3 &= \sum_{n=0}^{\infty} a'_3(n)q^n := \sum_{n=1}^{\infty} \frac{q^{30n^2-5n}}{1-q^{20n-2}} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{30n^2-5n+k(20n-2)}, \\
T'_4 &= \sum_{n=0}^{\infty} a'_4(n)q^n := \sum_{n=1}^{\infty} \frac{q^{30n^2-25n+4}}{1-q^{20n-12}} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{30n^2-25n+4+k(20n-12)}.
\end{aligned}$$

Let $T'_1 + T'_2 + T'_3 + T'_4 = \sum_{n=1}^{\infty} a'(n)q^n$, and let

$$\frac{2qJ_{10}^6}{J_{2,10}^2 J_{3,10}^2} = \sum_{n=1}^{\infty} b'(n)q^n.$$

We will show that $b'(n) > a'(n)$ for all $n \geq 1$.

Arguing as in Section 3.1, we see that for any $N \geq 1$, $a'_1(N), a'_2(N) \leq \lfloor \sqrt{\frac{N}{30}} \rfloor + 1$. Also, we note that since $30n^2 - 5n + k(20n - 2) > (5n)^2$ for all $n \geq 1$, we have that $a'_3(N) \leq \lfloor \sqrt{\frac{N}{25}} \rfloor$. Similarly, since $30n^2 - 25n + 4 > 30(n - 1)^2$ for all $n \geq 1$, we have that $a'_4(N) \leq \lfloor \sqrt{\frac{N}{30}} \rfloor + 1$. Together, noting that none of the T'_i have a constant term, we see that for any $n \geq 1$,

$$(18) \quad a'(n) \leq 3 \left\lfloor \sqrt{\frac{n}{30}} \right\rfloor + \left\lfloor \sqrt{\frac{n}{25}} \right\rfloor + 3.$$

We now examine the product. By (10), (11), and (12), we see that

$$\begin{aligned}
\frac{2qJ_{10}^6}{J_{2,10}^2 J_{3,10}^2} &= 2qL_{3,10} \left(\frac{(-q^2, -q^8; q^{10})_{\infty} (q^{10}; q^{10})_{\infty}}{(q^4, q^{16}; q^{20})_{\infty} (q^2, q^3, q^7, q^8; q^{10})_{\infty}} \right) \\
&= \frac{2q}{(1-q^2)(1-q^3)} \left(\frac{L_{3,10}}{(q^4, q^{16}; q^{20})_{\infty} (q^7, q^8, q^{12}, q^{13}; q^{10})_{\infty}} \cdot \sum_{n=-\infty}^{\infty} q^{5n^2+2n} \right).
\end{aligned}$$

Arguing as before, we find that the coefficient of q^n in $\frac{2q}{(1-q^2)(1-q^3)}$ is at least $2 \lfloor \frac{n-1}{6} \rfloor$ for $n \geq 1$. We see that $L_{3,10}$ has a constant term of 1, and by Lemma 2.1, $L_{3,10}$ has all nonnegative coefficients. Thus we have that for all $n \geq 1$,

$$b'(n) \geq 2 \left\lfloor \frac{n-1}{6} \right\rfloor.$$

Since $2 \lfloor \frac{n-1}{6} \rfloor > 3 \lfloor \sqrt{\frac{n}{30}} \rfloor + \lfloor \sqrt{\frac{n}{25}} \rfloor + 3$ for $n \geq 24$, it thus suffices to check that $b'(n) > a'(n)$ for $1 \leq n \leq 23$. A quick computation in Maple verifies that this is true.

3.3. Proof of (5). In order to prove (5), we need to show that

$$\sum_{n=0}^{\infty} (N_2(0, 10, 5n) + N_2(1, 10, 5n) - N_2(4, 10, 5n) - N_2(5, 10, 5n))q^n$$

has positive coefficients. Using the Theorem 2.3, we obtain that

$$(19) \quad \sum_{n=0}^{\infty} (N_2(0, 10, 5n) + N_2(1, 10, 5n) - N_2(4, 10, 5n) - N_2(5, 10, 5n))q^n$$

$$= \frac{1}{J_{5,20}} \left(\frac{2qJ_{20}^{15}J_{2,20}J_{10,20}}{J_{1,20}^3J_{3,20}^2J_{4,20}^2J_{5,20}^2J_{6,20}J_{7,20}^2J_{9,20}^3} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{10n^2+5n}}{1+q^{10n+2}} \right).$$

Since $\frac{1}{J_{5,20}}$ has all nonnegative coefficients and a constant term of 1, it suffices to show that

$$\frac{2qJ_{20}^{15}J_{2,20}J_{10,20}}{J_{1,20}^3J_{3,20}^2J_{4,20}^2J_{5,20}^2J_{6,20}J_{7,20}^2J_{9,20}^3} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{10n^2+5n}}{1+q^{10n+2}}$$

has all positive coefficients. Splitting up the sum as we do in the proof of (3), we obtain that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{10n^2+5n}}{1+q^{10n+2}} = S'' - T_1'' - T_2'' - T_3'' - T_4'',$$

where

$$S'' := \sum_{n=0}^{\infty} \frac{q^{40n^2+10n}}{1-q^{40n+4}} + \sum_{n=0}^{\infty} \frac{q^{40n^2+70n+27}}{1-q^{40n+24}} + \sum_{n=1}^{\infty} \frac{q^{40n^2+10n-2}}{1-q^{40n-4}} + \sum_{n=1}^{\infty} \frac{q^{40n^2-10n-9}}{1-q^{40n-24}},$$

and

$$\begin{aligned} T_1'' &= \sum_{n=0}^{\infty} a_1''(n)q^n := \sum_{n=0}^{\infty} \frac{q^{40n^2+30n+2}}{1-q^{40n+4}} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q^{40n^2+30n+2+k(40n+4)}, \\ T_2'' &= \sum_{n=0}^{\infty} a_2''(n)q^n := \sum_{n=0}^{\infty} \frac{q^{40n^2+50n+15}}{1-q^{40n+24}} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q^{40n^2+50n+15+k(40n+24)}, \\ T_3'' &= \sum_{n=0}^{\infty} a_3''(n)q^n := \sum_{n=1}^{\infty} \frac{q^{40n^2+30n-4}}{1-q^{40n-4}} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{40n^2+30n-4+k(40n-4)}, \\ T_4'' &= \sum_{n=0}^{\infty} a_4''(n)q^n := \sum_{n=1}^{\infty} \frac{q^{40n^2-30n+3}}{1-q^{40n-24}} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{40n^2-30n+3+k(40n-24)}. \end{aligned}$$

We see that S'', T_1'', \dots, T_4'' all have nonnegative coefficients. Thus to prove (5), it suffices to show that

$$\frac{2qJ_{20}^{15}J_{2,20}J_{10,20}}{J_{1,20}^3J_{3,20}^2J_{4,20}^2J_{5,20}^2J_{6,20}J_{7,20}^2J_{9,20}^3} - T_1'' - T_2'' - T_3'' - T_4''$$

has positive coefficients. Let $T_1'' + T_2'' + T_3'' + T_4'' = \sum_{n=1}^{\infty} a''(n)q^n$, and let

$$\frac{2qJ_{20}^{15}J_{2,20}J_{10,20}}{J_{1,20}^3J_{3,20}^2J_{4,20}^2J_{5,20}^2J_{6,20}J_{7,20}^2J_{9,20}^3} = \sum_{n=1}^{\infty} b''(n)q^n.$$

We will show that $b''(n) > a''(n)$ for all $n \geq 1$.

Again arguing as in Section 3.1, we see that for any $N \geq 1$, $a_1''(N), a_2''(N) \leq \lfloor \sqrt{\frac{N}{40}} \rfloor + 1$, and $a_3''(N) \leq \lfloor \sqrt{\frac{N}{40}} \rfloor$. Also, since $40n^2 - 30n + 3 > 6(2n-1)^2$ for all $n \geq 1$, we have that $a_4''(N) \leq \lfloor \sqrt{\frac{N}{24}} \rfloor + 1$. Together, noting that none of the T_i'' have a constant term, we see that for any $n \geq 1$,

$$(20) \quad a''(n) \leq 3 \left\lfloor \sqrt{\frac{n}{40}} \right\rfloor + \left\lfloor \sqrt{\frac{n}{24}} \right\rfloor + 3.$$

We now examine the product. By (10), we find that

$$\frac{2qJ_{2,20}J_{10,20}J_{20}^{15}}{J_{6,20}J_{3,20}^2J_{4,20}^2J_{5,20}^2J_{7,20}^2J_{1,20}^3J_{9,20}^3} = \frac{2q}{(1-q)^2}L_{9,20}^2 \left(\frac{(-q, -q^9, -q^{11}, -q^{19}; q^{20})_{\infty}(-q^5, -q^{15}; q^{20})_{\infty}^2}{(q^6, q^{14}; q^{20})_{\infty}(q^3, q^4, q^7, q^{13}, q^{16}, q^{17}, q^{19}, q^{21}; q^{20})_{\infty}^2(q^{19}; q^{20})_{\infty}^3} \right).$$

Expanding gives that $\frac{2q}{(1-q)^2} = \sum_{n=1}^{\infty} 2nq^n$. We know by Lemma 2.1 that $L_{9,20}$ has nonnegative coefficients and a constant term of 1, and we can observe that this is true for the rest of the product as well. Thus, we have for all $n \geq 1$,

$$b''(n) \geq 2n.$$

Since $2n > 3 \lfloor \sqrt{\frac{n}{40}} \rfloor + \lfloor \sqrt{\frac{n}{24}} \rfloor + 3$ for $n \geq 2$, it thus suffices to observe that $2 = b''(1) > a''(1) = 0$.

3.4. Proof of (6). In order to prove (6), we need to show that

$$\sum_{n=0}^{\infty} (N_2(0, 10, 5n+4) + N_2(1, 10, 5n+4) - N_2(4, 10, 5n+4) - N_2(5, 10, 5n+4))q^n$$

has positive coefficients. Using the Theorem 2.3, we obtain that

$$(21) \quad \sum_{n=0}^{\infty} (N_2(0, 10, 5n+4) + N_2(1, 10, 5n+4) - N_2(4, 10, 5n+4) - N_2(5, 10, 5n+4))q^n = \frac{1}{J_{5,20}} \left(\frac{2J_{20}^{15}J_{6,20}J_{10,20}}{J_{1,20}^2J_{2,20}J_{3,20}^3J_{5,20}^2J_{7,20}^3J_{8,20}^2J_{9,20}^2} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{10n^2+15n+4}}{1+q^{10n+6}} \right).$$

Since $\frac{1}{J_{5,20}}$ has all nonnegative coefficients and a constant term of 1, it suffices to show that

$$\frac{2J_{20}^{15}J_{6,20}J_{10,20}}{J_{1,20}^2J_{2,20}J_{3,20}^3J_{5,20}^2J_{7,20}^3J_{8,20}^2J_{9,20}^2} + \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{10n^2+15n+4}}{1+q^{10n+6}}$$

has all positive coefficients. Splitting up the sum as we do in the proof of (3), we obtain that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{10n^2+15n+4}}{1+q^{10n+6}} = S''' - T_1''' - T_2''' - T_3''' - T_4''',$$

where

$$S''' := \sum_{n=0}^{\infty} \frac{q^{40n^2+30n+4}}{1-q^{40n+12}} + \sum_{n=0}^{\infty} \frac{q^{40n^2+90n+45}}{1-q^{40n+32}} + \sum_{n=1}^{\infty} \frac{q^{40n^2-10n-2}}{1-q^{40n-12}} + \sum_{n=1}^{\infty} \frac{q^{40n^2-30n-3}}{1-q^{40n-32}},$$

and

$$\begin{aligned} T_1''' &= \sum_{n=0}^{\infty} a_1'''(n)q^n := \sum_{n=0}^{\infty} \frac{q^{40n^2+50n+10}}{1-q^{40n+12}} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q^{40n^2+50n+10+k(40n+12)}, \\ T_2''' &= \sum_{n=0}^{\infty} a_2'''(n)q^n := \sum_{n=0}^{\infty} \frac{q^{40n^2+70n+29}}{1-q^{40n+32}} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q^{40n^2+70n+29+k(40n+32)}, \\ T_3''' &= \sum_{n=0}^{\infty} a_3'''(n)q^n := \sum_{n=1}^{\infty} \frac{q^{40n^2+10n-8}}{1-q^{40n-12}} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{40n^2+10n-8+k(40n-12)}, \\ T_4''' &= \sum_{n=0}^{\infty} a_4'''(n)q^n := \sum_{n=1}^{\infty} \frac{q^{40n^2-50n+13}}{1-q^{40n-32}} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} q^{40n^2-50n+13+k(40n-32)}. \end{aligned}$$

We see that $S''', T_1''', \dots, T_4'''$ all have nonnegative coefficients. Thus to prove (6), it suffices to show that

$$\frac{2J_{20}^{15}J_{6,20}J_{10,20}}{J_{1,20}^2J_{2,20}J_{3,20}^3J_{5,20}^2J_{7,20}^3J_{8,20}^2J_{9,20}^2} - T_1''' - T_2''' - T_3''' - T_4'''$$

has positive coefficients. Let $T_1''' + T_2''' + T_3''' + T_4''' = \sum_{n=1}^{\infty} a'''(n)q^n$, and let

$$\frac{2J_{20}^{15}J_{6,20}J_{10,20}}{J_{1,20}^2J_{2,20}J_{3,20}^3J_{5,20}^2J_{7,20}^3J_{8,20}^2J_{9,20}^2} = 2 + \sum_{n=1}^{\infty} b'''(n)q^n.$$

We will show that $b'''(n) > a'''(n)$ for all $n \geq 1$.

Again arguing as in Section 3.1, we see that for any $N \geq 1$, $a_1'''(N), a_2'''(N) \leq \lfloor \sqrt{\frac{N}{40}} \rfloor + 1$, and $a_3'''(N) \leq \lfloor \sqrt{\frac{N}{40}} \rfloor$. Also, since $40n^2 - 50n + 13 > 40(n-1)^2$ for all $n \geq 1$, we have that $a_4'''(N) \leq \lfloor \sqrt{\frac{N}{40}} \rfloor + 1$. Together, noting that none of the T_i''' have a constant term, we see that for any $n \geq 1$,

$$(22) \quad a'''(n) \leq 4 \left\lfloor \sqrt{\frac{n}{40}} \right\rfloor + 3.$$

By computing the $n = 0$ term of the sum appearing in (6), we find that the constant term is 1. We may thus instead consider

$$(23) \quad \frac{2J_{6,20}J_{10,20}J_{20}^{15}}{J_{2,20}J_{1,20}^2J_{5,20}^2J_{8,20}^2J_{9,20}^3J_{3,20}^3J_{7,20}^3} - \frac{1}{(1-q)^2}.$$

We now examine the product. By (10), we find that

$$\begin{aligned} & \frac{2J_{6,20}J_{10,20}J_{20}^{15}}{J_{2,20}J_{1,20}^2J_{5,20}^2J_{8,20}^2J_{9,20}^3J_{3,20}^3J_{7,20}^3} \\ &= \frac{2}{(1-q)^2} L_{9,20}^2 \left(\frac{(-q^3, -q^7, -q^{13}, -q^{17}; q^{20})_{\infty} (-q^5, -q^{15}; q^{20})_{\infty}^2}{(q^2, q^{18}; q^{20})_{\infty} (q^3, q^7, q^8, q^{12}, q^{13}, q^{17}, q^{19}, q^{21}; q^{20})_{\infty}^2} \right). \end{aligned}$$

As in Section 3.3, we have that expanding gives that $\frac{2}{(1-q)^2} = \sum_{n=0}^{\infty} 2(n+1)q^n$. Also, we have already seen that $L_{9,20}$ has nonnegative coefficients and a constant term of 1, and we can observe that this is true for the rest of the product as well. Thus, we have for all $n \geq 1$,

$$b'''(n) \geq 2(n+1).$$

Since $2(n+1) > 4 \lfloor \sqrt{\frac{n}{40}} \rfloor + 3$ for $n \geq 1$, this completes the proof of (6).

REFERENCES

- [1] George E Andrews. *The theory of partitions*. Number 2. Cambridge university press, 1998.
- [2] A. O. L. Atkin and P. Swinnerton-Dyer. Some properties of partitions. *Proc. London Math. Soc. (3)*, 4:84–106, 1954.
- [3] Freeman J Dyson. Some guesses in the theory of partitions. *Eureka (Cambridge)*, 8(10), 1944.
- [4] Jeremy Lovejoy and Robert Osburn. Rank differences for overpartitions. *Q. J. Math.*, 59(2):257–273, 2008.
- [5] Jeremy Lovejoy and Robert Osburn. M_2 -rank differences for partitions without repeated odd parts. *J. Théor. Nombres Bordeaux*, 21(2):313–334, 2009.
- [6] Jeremy Lovejoy and Robert Osburn. M_2 -rank differences for overpartitions. *Acta Arith.*, 144(2):193–212, 2010.
- [7] Renrong Mao. Ranks of partitions modulo 10. *J. Number Theory*, 133(11):3678–3702, 2013.
- [8] Renrong Mao. The M_2 -rank of partitions without repeated odd parts modulo 6 and 10. *Ramanujan J.*, 37(2):391–419, 2015.

EMORY UNIVERSITY

E-mail address: `ealwais@emory.edu`

VASSAR COLLEGE

E-mail address: `eliannuzzi@vassar.edu`

DEPARTMENT OF MATHEMATICS, OREGON STATE UNIVERSITY, 368 KIDDER HALL, CORVALLIS, OR 97331

E-mail address: `swisherh@math.oregonstate.edu`